CFT/TFT correspondence beyond semisimplicity

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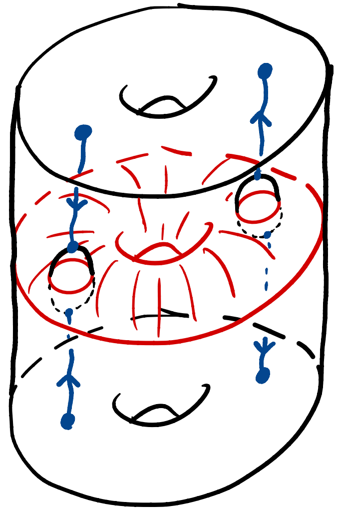
Aaron Hofer

Max-Planck-Institute for Mathematics, Bonn

joint work with Ingo Runkel

based on [2405.18038] & work in progress



Outline:

i) Motivation

ii) 2d CFTs

iii) 3d defect TFTs

iv) FRS-construction

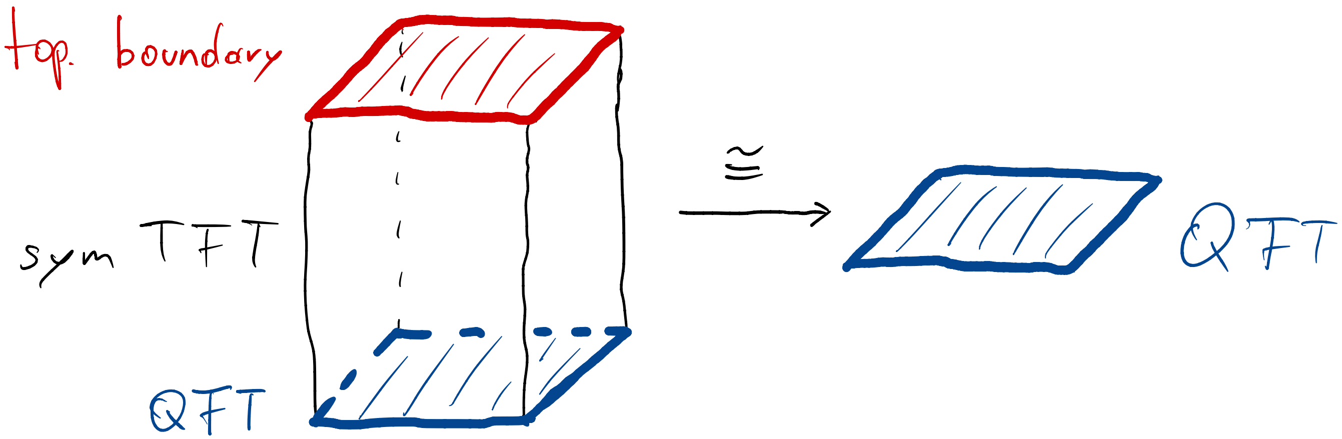
v) FRS-construction 2.0

i) Motivation

Problem: Quantum field theories are generally very hard to study "systematically".

Idea: Use symmetries to make problem approachable!

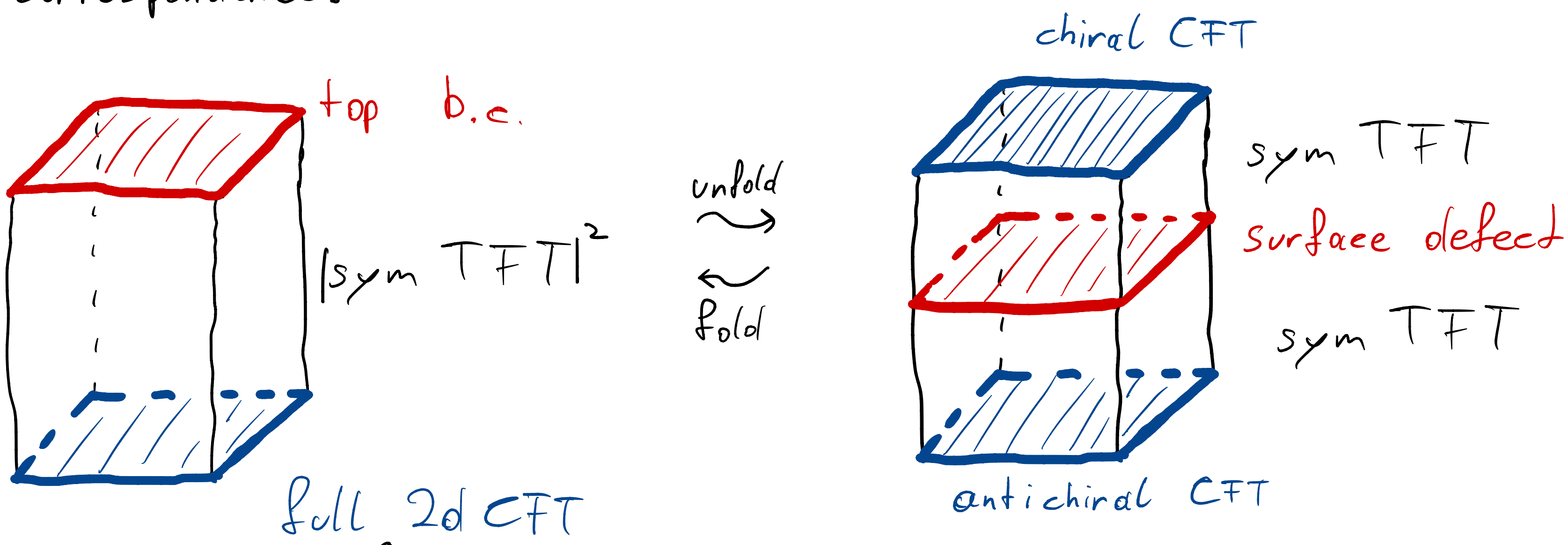
"Symmetry topological field theory framework"



Apply these ideas to study full 2d conformal field theories

i) Motivation

CFT–TFT correspondence:



What is this mathematically?

chiral conformal block space $\cong$ TFT state space

full 2d CFT: vertex operator algebra $\longrightarrow$ sym. TFT

+ modular functor $\xleftarrow{\text{conjecture}}$ sym. TFT

+ field content

+ correlators $\xleftarrow{\text{FRS-construction}}$ TFT + top. boundary condition / surface defect

ii) 2d CFT's

For 2d CFT's we expect the following mathematical structure:

1) chiral symmetry algebras are described as vertex operator algebras (VOA's)

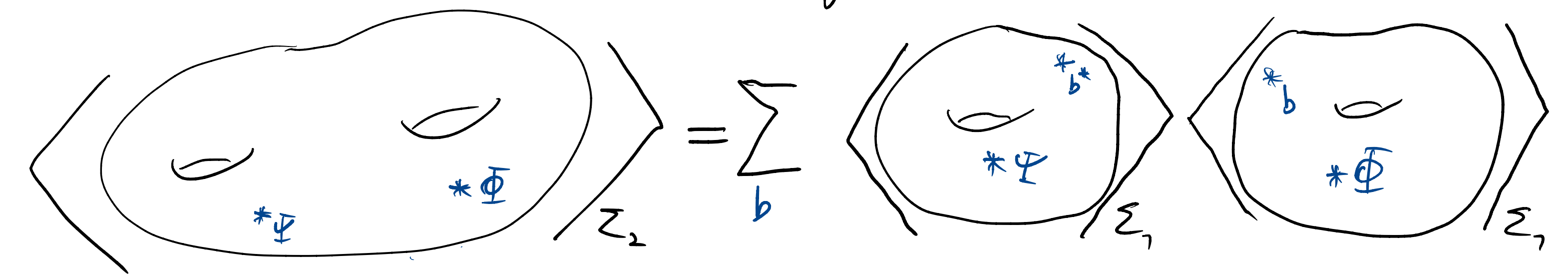
2) for any complex curve Σ^c we get a conformal block space $\Sigma^c \longmapsto BL(\Sigma^c)$

3) fields $\in \mathbb{F} \in$ VOA-modules, which assemble into "nice" category $Rep(V)$

4) correlators of full CFT are elements of block space of Schottky double

$$Cor(\Sigma) \in BL(\hat{\Sigma})$$

5) nice behaviour under cutting of surfaces (factorisation)

"


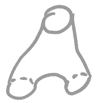
"

6) correlators should be invariant under mapping class group actions (related to single valuednes)

ii) 2d CFT's

This can be packaged neatly using (higher) categorical language:

Def A (topological) modular functor is a symmetric monoidal 2-functor

	$Bl : \mathrm{Bord}_{2+\varepsilon,2,1}$	$\longrightarrow \mathcal{L}ex_{\mathbb{K}}$	CFT interpretation:
0:	circles 	finite $\mathbb{K}$-linear ab. cats	$\mathrm{Rep}(V)$
1:	bordisms 	left exact functors $\mathcal{A} \longrightarrow \mathcal{B}$	$Bl(\Sigma)$
2:	diffeo's / isotopy	natural traf.'s	MCG-action on Bl
$\diamond$:	gluing	horizontal composition	factorisation
$\circ$:	composition	vertical composition	
$\square$:	disjoint union	Deligne $\boxtimes$	

Rem There is also an algebro-geometric version of modular functors and it is conjectured to be equivalent to the topological one given above under certain conditions.

ii) 2d CFT's

Def A *full CFT* for a modular functor

$$\mathcal{B}l : \mathrm{Bord}_{2+\varepsilon,2,1} \longrightarrow \mathcal{L}ex_{\mathbb{K}}$$

is a braided monoidal oplax natural transformation

$$\begin{tikzcd} \mathrm{Bord}_{2+\varepsilon,2,1} \arrow[r, bend left=50, "\Delta_{\mathbb{K}}"{name=U}] \arrow[r, bend right=50, "\mathcal{B}l \circ (\hat{-})"'{name=D}] & \mathcal{L}ex_{\mathbb{K}} \arrow[from=U, to=D, phantom, "\Downarrow \mathrm{Cor}"] \end{tikzcd}$$

Orientation double functor $\hat{M} := M \sqcup -M$

where $\Delta_{\mathbb{K}} : \mathrm{Bord}_{2+\varepsilon,2,1} \longrightarrow \mathcal{L}ex_{\mathbb{K}}$ is the constant 2-functor to $\mathrm{vect}_{\mathbb{K}}$.

This definition encodes:

i) For every $\Gamma \in \mathrm{Bord}_{2+\varepsilon,2,1}$ (1-manifold) a left exact functor:

$$\mathrm{Cor}_{\Gamma}(-) : \mathcal{B}l(\Gamma) \longrightarrow \mathrm{vect} \cong \mathrm{Hom}_{\mathcal{B}l(\Gamma)}(\mathbb{F}_{\Gamma}, -) \longrightarrow \mathbb{F}_{S^1}$$

state space on S^1 (field content)

ii) For every 1-morphism $\Gamma \xrightarrow{\Sigma} \Gamma'$ (surface) in $\mathrm{Bord}_{2+\varepsilon,2,1}$ a natural transformation:

$$\mathrm{Cor}_{\Sigma} \in \mathrm{Nat}\left(\mathrm{cor}_{\Gamma'} \circ \Delta_{\mathbb{K}}(\Sigma), \mathcal{B}l(\hat{\Sigma}) \circ \mathrm{cor}_{\Gamma}\right) \cong \mathcal{B}l(\hat{\Sigma})(\mathbb{F}_{\Gamma}; \mathbb{F}_{\Gamma'}) \longrightarrow \text{Correlators}$$

ii) 2d CFT's

Def A full CFT for a modular functor

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This definition encodes:

iii) Naturality axioms encode mapping class group covariance and factorisation of correlators.

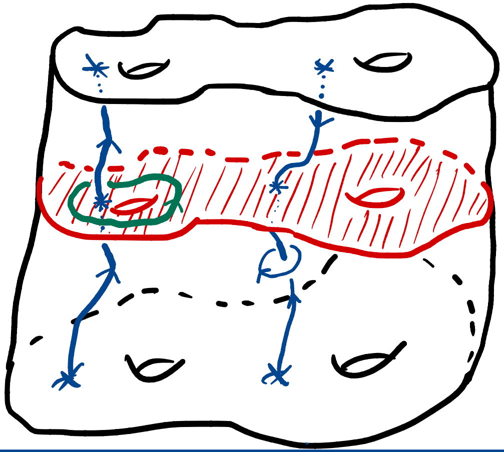
(2-morphism naturality)

(1-morphism naturality)

+ . . .

iii) 3d defect TFT's

There is a notion of TFT with defects, where submanifolds of various codimension are decorated using extra data.



Thm. [Reshetikhin-Turaev, Carqueville-Schaumann-Runkel, Koppen-Mulevičius-Schweigert-Runkel]
Let $\mathcal{C}$ be a modular fusion cat. There exists a TFT with defects

$$Z_{\mathcal{C}} : \mathrm{Bord}^{def}_{3,2}(\mathbb{D}_{\mathcal{C}}) \longrightarrow \mathrm{Vect}$$

constructed from $\mathcal{C}$.

Rem. The TFT $Z_{\mathcal{C}}$ induces a modular functor $Bl_{\mathcal{C}}$. For $\mathcal{C} = \mathrm{Rep}(\mathcal{V})$ with $\mathcal{V}$ a rational VOA it is conjectured that $Bl_{\mathcal{C}} \cong Bl_{\mathcal{V}}$.

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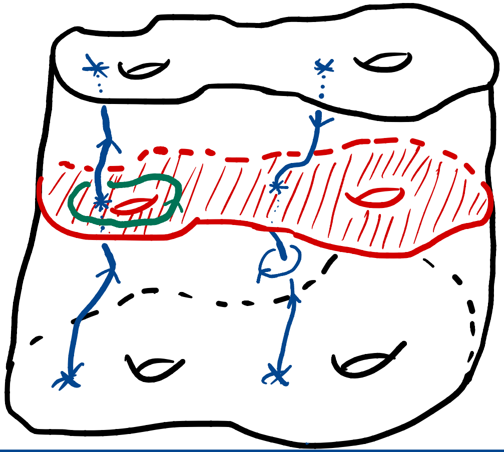

Thm. [Reshetikhin-Turaev, Carqueville-Schaumann-Runkel, Koppen-Mulevičius-Schweigert-Runkel, De-Renzi-Gainutdinov-Geer-Patureau-Mirand-Runkel, H-Runkel]
Let $\mathcal{C}$ be a modular tensor cat. There exists a TFT with defects

$$Z_{\mathcal{C}} : \mathrm{Bord}_{3,2}^{\mathrm{def},n.c}(\mathbb{D}_{\mathcal{C}}) \longrightarrow \mathrm{Vect}$$

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Rem. The TFT $Z_{\mathcal{C}}$ induces a modular functor $Bl_{\mathcal{C}}$. For $\mathcal{C} = \mathrm{Rep}(V)$ with V a finite Cog. VOA it is conjectured that $Bl_{\mathcal{C}} \cong Bl_V$.

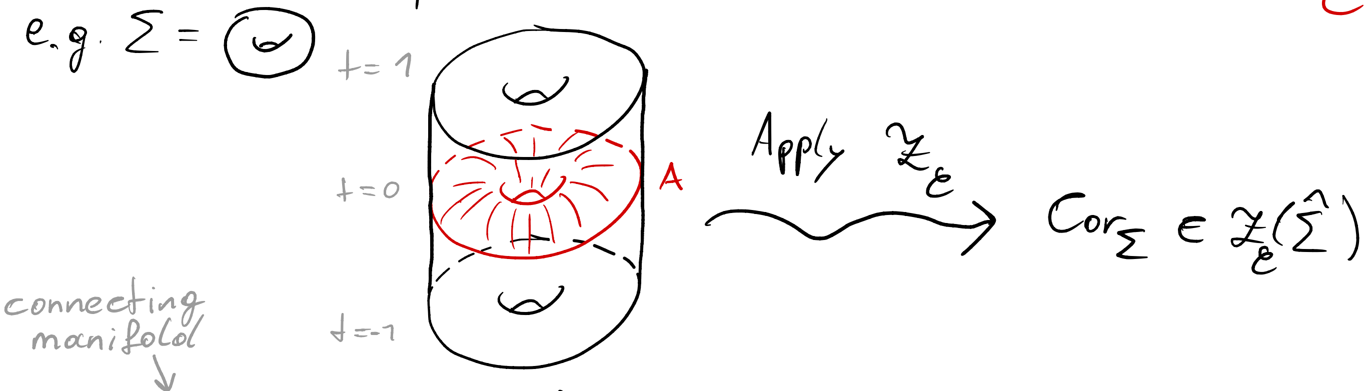
iv) FRS-construction

For Σ closed surface: $\mathrm{Cor}_\Sigma \in Bl_{\mathcal{C}}(\hat{\Sigma}) \cong Z_{\mathcal{C}}(\hat{\Sigma})$ with $\hat{\Sigma} = \Sigma \sqcup -\Sigma$

<u>Main idea:</u> Can we find a bordism $\emptyset \xrightarrow{M_\Sigma} \hat{\Sigma}$ such that $Z_{\mathcal{C}}(M_\Sigma)$ satisfies the conditions of a correlator?

Yes! But not uniquely, need surface defect $A \in D^2_{\mathcal{C}} \subset D_{\mathcal{C}}$ as extra input.

e.g. $\Sigma =$



$M_\Sigma = \Sigma \times [-1,1]$ with surface defect A at $\Sigma \times \{0\}$.

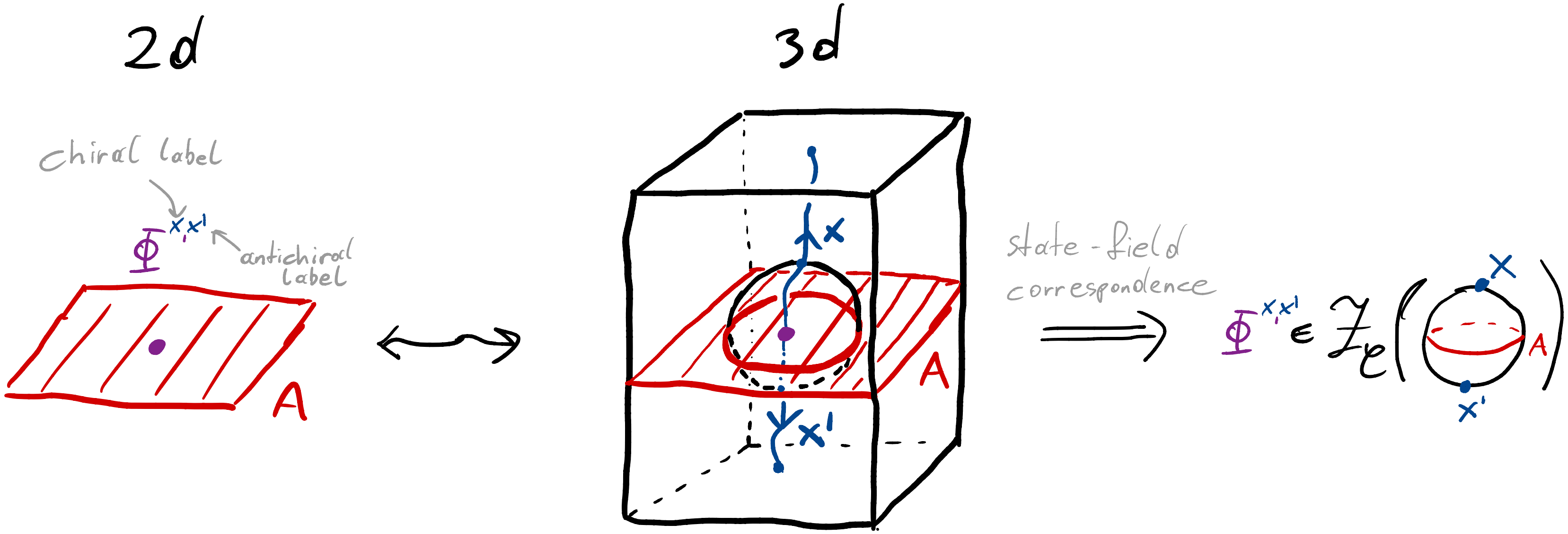
<u>Thm</u> [Fuchs-Runkel-Schweigert]
For $\mathcal{C}$ fusion, $\mathrm{Cor}^A_\Sigma = Z_{\mathcal{C}}(M^A_\Sigma)$ gives consistent system of correlators. (2-morphism level of $\mathrm{Cor}: \Delta_{\mathbb{K}} \Rightarrow Bl_{\mathcal{C}} \circ (\hat{-})$)

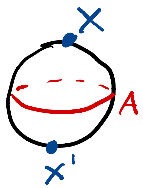
(1-morphism level of Cor)

<u>Q's</u> 1) In [FRS] the field content is determined algebraically can we get it topologically as well?

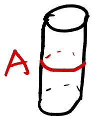
2) What about non-semisimple $\mathcal{C}$?

v) FRS-construction 2.0





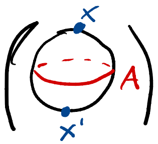
comes from the connecting manifold $M^{A}_{S^1}$ of S^1



$: \emptyset \longrightarrow \hat{S}^1 = S^1 \sqcup -S^1$

In analogy to construction of $Bl_{\mathcal{C}}$ on surfaces, we get a functor from $M^{A}_{S^1}$:

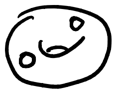
$$Cor^{A}_{S^1} : Bl_{\mathcal{C}}(S^1) = \mathcal{C} \boxtimes \mathcal{C} \longrightarrow vect_{\mathbb{K}}$$

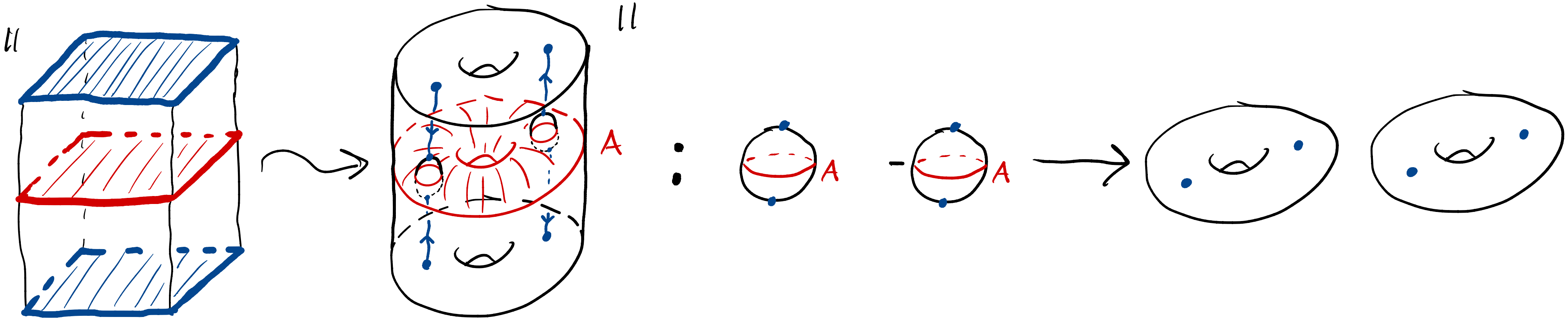
$$x \boxtimes x' \longmapsto \mathcal{Z}_{\mathcal{C}}\Big($$  $$\Big)$$

$F^{A}_{S^1} \in \mathcal{C} \boxtimes \mathcal{C}$ is representing the space of CFT bulk fields!

$\simeq Rep(V \otimes V)$

v) FRS-construction 2.0

Back to correlators: $\Sigma =$  $: S^1 \to S^1$



$$\mathcal{Z}_{\mathcal{C}}\left(M_{\Sigma}^{A\ldots}\right) : \mathcal{Z}_{\mathcal{C}}\left(M_{S^1}^{A\ldots}\right) \otimes_{\mathbb{K}} \mathcal{Z}_{\mathcal{C}}\left(-M_{S^1}^{A\ldots}\right) \longrightarrow \mathcal{Z}_{\mathcal{C}}\left(\hat{\Sigma}^{\ldots}\right)$$

natural in
chiral & antichiral
labels

$$\Longrightarrow \qquad Cor_{\Sigma}^{A} : Cor_{S^1}^{A} \otimes_{\mathbb{K}} Cor_{S^1}^{A+} \Longrightarrow Bl_{\mathcal{C}}(\hat{\Sigma})$$

$$\Longleftrightarrow \qquad Cor_{S^1}^{A} \diamond \Delta_{\mathbb{K}}(\Sigma) \Longrightarrow Bl_{\mathcal{C}}(\Sigma) \diamond Cor_{S^1}^{A}$$

v) FRS-construction 2.0

Thm [Fuchs–Runkel–Schweigert, H–Runkel]
Let $\mathcal{C}$ be a modular tensor cat. Under some technical assumptions on $\mathcal{Z}_{\mathcal{C}}$, evaluation of the connecting manifold gives a full CFT

$$\begin{tikzcd} \mathrm{Bord}_{2+\varepsilon,2,1} \arrow[r, bend left, "\Delta_{\mathbb{K}}"] \arrow[r, bend right, "\widehat{BL_{\mathcal{C}}\circ(-)}"'] \arrow[r, phantom, "\Downarrow \mathrm{Cor}^{A}"] & \mathcal{L}ex_{\mathbb{K}} \end{tikzcd}.$$

for any $A \in D^2_{\mathcal{C}}$.

Rem's i) For $\mathcal{C}$ semisimple we recover [FRS].

ii) For $\mathcal{C}$ non-semisimple and $A = \mathbb{1}$ (transparent surface defect) we reproduce and extend results of [Fuchs–Gannon–Schaumann–Schweigert].

Outlook

- Computations with A non-trivial?
- More general surface defects in Z_e?
- Computations for specific input cats, e.g. $\mathrm{Rep}(W_p) \simeq \mathrm{Rep}(u_q(sl_2))$
- Relation to other approaches?
- Relation to skein theory with defects?